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ASSIGNMENT GAMES AND PERMUTATION GAMES

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ABSTRACT: Relations between permutation games and assignment games are studied. Such games are totally balanced and have price equilibria. Multidimensional permutation games and assignment games are introduced. Examples show that such games may have an empty core. Special classes of multidimensional permutation games and assignment games are defined for which the members are totally balanced.

1. INTRODUCTION

In this paper we consider properties of and relations between assignment games and permutation games. Furthermore, extensions of such games are studied.

Assignment games were introduced by Shapley and Shubik [5]. They consider a situation with two different types of agents, buyers and sellers, the latter hereafter to be called merchants. Each merchant possesses one indivisible good, e.g. a house, which he wants to sell and each buyer wants to buy one house. Sidepayments between all players are allowed. Let B be the set of buyers and M the set of merchants. For every $j \in M$ we denote the value for him of his indivisible good by c_j. For all $i \in B$, $j \in M$, h_{ij} denotes the value for buyer i of the indivisible good of merchant j. For all $i \in B$, $j \in M$ we define:

$$a_{ij} := \max \{0, h_{ij} - c_j\}.$$

The assignment game $\langle B \cup M, v\rangle$ with revenues a_{ij} is defined by:

$$v(S) := \max(a_{i_1 j_1} + a_{i_2 j_2} + \dots + a_{i_r j_r}) \text{ for all } S \in 2^N,$$

where $r = \min\{|S \cap B|, |S \cap M|\}$ and the maximum is taken over all assignments of players $j_1, \dots, j_r$ in $S \cap M$ to players $i_1, \dots, i_r$ in $S \cap B$. Here and in the rest of this paper we take the maximum (minimum) over the empty set to be equal to zero. This implies that $v(S)=0$ for $S \subset M$ or $S \subset B$. Shapley and Shubik [5] prove with the aid of the duality theorem for linear programming and the theorem of Birkhoff - von Neumann, which states that the permutation matrices are the extreme points of the set of doubly stochastic matrices, that the

assignment game has a non-empty core. A doubly stochastic $n\times n$-matrix $[d_{ij}]_{i=1,j=1}^{n\ \ n}$ is an $n\times n$-matrix such that for all $i,j \in N$: $d_{ij} \geq 0$, $\sum_j d_{ij} = 1$, $\sum_i d_{ij} = 1$.
An $n\times n$-permutation matrix $[p_{ij}]_{i=1,j=1}^{n\ \ n}$ is a doubly stochastic matrix with all entries p_{ij} equal to zero or one.
Shapley and Shubik [5] also show that every core element is associated with a price mechanism.

Permutation games were introduced by Tijs, Parthasarathy, Potters and Rajendra Prassad [6]. They describe a situation in which n persons all have one job to be processed and one machine on which each job can be processed. No machine is allowed to process more than one job. Sidepayments between the players are allowed. If player i processes his job on the machine of player j the processing costs are c_{ij}. Let $N = \{1,\dots,n\}$ be the set of players. The permutation game $\langle N,c\rangle$ with costs c_{ij} is defined by

$$c(S) := \min_{\pi_S \in \Pi_S} \sum_{i \in S} c_{i\pi(i)} \quad \text{for all } S \in 2^N.$$

Here Π_S is the class of all S-permutations. Tijs et al. [6] prove with the aid of the Bondareva-Shapley theorem on balanced games and the Birkhoff-von Neumann theorem on doubly stochastic matrices that the core of a permutation game is not empty.

The organization of the rest of this paper is as follows. In section 2 we describe some relations between assignment games and permutation games. In section 3 these games are described in terms of exchange economies with indivisibilities. With the aid of a recent elegant theorem of Gale [2] on price equilibria in such economies we give another proof of the existence of price mechanisms for such games. In section 4 we consider generalizations of the assignment game and the permutation game to situations with two types of indivisible goods. The games we obtain can have empty cores. Special classes of such games for which the core is not empty are treated in section 5.

2. RELATIONS BETWEEN ASSIGNMENT GAMES AND PERMUTATION GAMES

In an assignment game the set of players is partitioned into two sets and only assignments between members of different sets will occur. The situation is typically bipartite. Generally in a permutation game such a partition cannot be made. But when the cost matrix has a certain form such a partition will arise naturally. In particular we can associate with every assignment game $\langle B \cup M, v\rangle$ a permutation game $\langle B \cup M, c\rangle$ such that $c(S) = -v(S)$ for all $S \in 2^N$. In the following we will denote $|B|$ by b and $|M|$ by m. Let $A = [a_{ij}]_{i=1,j=1}^{b\ \ m}$ be the revenue matrix of $\langle B \cup M, v\rangle$. Let $\langle B \cup M, c\rangle$ be the permuation game with $(b+m)\times(b+m)$-cost matrix $\begin{bmatrix} \Theta_1 & -A \\ \Theta_2 & \Theta_3 \end{bmatrix}$, where the first b rows (columns) correspond to the buyers and the other m rows (columns) to the merchants. Hence, the three submatrices Θ_1, Θ_2 and Θ_3 are the $b\times b$-, the $m\times b$- and the $m\times m$-zero matrix, respectively. Then $c(S) = -v(S)$ for every $S \in 2^N$ and an optimal permutation, in the game c corresponds to an optimal assignment in v. Further it is clear that we have $C(c) = -C(v)$ where

$$C(c) := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = c(N),\ \sum_{i\in S} x_i \le c(S) \text{ for each } S \in 2^N\},$$

$$C(v) := \{x \in \mathbb{R}^{b+m} : \sum_{i\in B\cup M} x_i = v(B\cup M),\ \sum_{i\in S} x_i \ge v(S) \text{ for each } S \in 2^{B\cup M}\}$$

are the <u>core</u> of the game c and the game v, respectively. Hence, assignment games with b buyers and m merchants can be seen as $(b+m)\times(b+m)$-permutation games.

The question arises if for every permutation game $\langle N,c\rangle$ we can construct an assignment game $\langle N,v\rangle$ such that $c(S) = -v(S)$ for every $S \in 2^N$. This is not the case. A necessary condition is that there exists a partition of N into two subsets N_1, N_2 such that $c(S) = 0$ whenever $S \subset N_1$ or $S \subset N_2$. But this condition is not sufficient as the following example shows.

<u>Example</u>. Let $N = \{1,2,3\}$ and let $\langle N,c\rangle$ be the permutation game with cost matrix $\begin{bmatrix} 0 & -2 & -1 \\ -1 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ Then $c(\{i\}) = 0$ for each $i \in N$

$c(\{1,2\}) = c(\{1,3\}) = -3$

$c(\{2,3\}) = 0$, $c(\{1,2,3\}) = -4$.

This game fulfils the above condition with $N_1 = \{1\}$ and $N_2 = \{2,3\}$ but the game $\langle N,v\rangle$ defined by $v(S) = -c(S)$ for all $S \in 2^N$ is not an assignment game as we can show as follows. If $\langle N,v\rangle$ were an assignment game, then $a_{12} = a_{13} = 3$ which would lead to the contradiction $4 = v(\{1,2,3\}) = \max\{a_{12}, a_{13}\} = 3$.

By giving another proof of the non-emptiness of the core of a permutation game, which is similar to the proof of Shapley and Shubik for the assignment game, another relation between the two games becomes clear. We consider the problem of determining c(N). That is the following integer programming problem

$$\min \sum_{i,j\in N} c_{ij}x_{ij} \quad \text{subject to}$$

$$\sum_{i\in N} x_{ij} = 1 \text{ for all } j \in N, \ \sum_{j\in N} x_{ij} = 1 \text{ for all } i \in N, \ x_{ij} \geq 0$$
$$\text{and } x_{ij} \text{ is integer for all } i,j \in N.$$

Because of the theorem of Birkhoff-von Neumann this problem is equivalent to the linear programming problem with the integer conditions left out. So c(N) is equal to the value of the dual problem which is

$$\max \sum_{i\in N} y_i + \sum_{i\in N} z_i \quad \text{subject to}$$
$$y_i+z_j \leq c_{ij} \text{ for all } i,j \in N.$$

Let $(y_1,\ldots,y_n,z_1,\ldots,z_n)$ be a solution of this dual problem then $\sum_{i\in N} (y_i+z_i) = c(N)$ and $\sum_{i\in S} (y_i+z_i) \leq \sum_{i\in S} c_{i\pi_S(i)}$ for all permutations π_S of S, so $\sum_{i\in S} (y_i+z_i) \leq c(S)$ for all $S \in 2^N$ and hence $(y_1+z_1,\ldots,y_n+z_n) \in C(c)$. Every player $i \in N$ can be seen as consisting of a buyer's part associated with y_i and a seller's part associated with z_i. A price mechanism for this core element is as follows. Let every player $j \in N$ ask a price $-z_j$ for his good. To minimize his costs player i will look for the minimum of $c_{ij}-z_j$ over all $j \in N$. All these numbers will be greater than or equal to y_i and the maximalization of the dual objective function ensures that at least one will be equal to y_i. So the costs of player i will be y_i minus the price he gets for his good, namely $-z_i$. His total costs are $y_i-(-z_i) = y_i+z_i$ which is what the core element we are considering allocates to him. A permutation may be interpreted as assigning a player's buyer's part to another player's (or his own) seller's part.

3. EXCHANGE ECONOMIES WITH INDIVISIBILITIES

Another way of looking at permutation games and assignment games is to consider them as exchange economies with indivisibilities and money. In Curiel [1] this is done for the permutation game using a model of Quinzii [3]. This results in another proof of the non-emptiness of the core of a permutation game. In this section we will use a model of Gale [2] concerning exchange economies with indivisibilities and money to prove the existence of a price equilibrium for the permutation game. In order to apply Gale's theorem to the permutation game we have to define for every $i,j \in N$ a subset C_j^i of $\mathbb{R}^n$. The meaning of this set is that for every price vector $p \in C_j^i$ player i will want to buy the machine of player j at price p_j and sell his machine at price p_i. Further we have to define for every $i \in N$ a subset C_0^i of $\mathbb{R}^n$ such that for every price vector $p \in C_0^i$ player i will sell his machine at price p_i and won't buy any machine of the other players but let his job remain unfinished and receive a penalty for that. This we do as follows. Let <N,c> be a permutation game with cost matrix $C = [c_{ij}]_{i=1,j=1}^{n\ \ n}$. Without loss of generality we assume that C is non-negative. Let $U > \max c_{ij}$. (U is the penalty a player gets when his job remains unfinished). For all $i,j \in N$ we define

$$C_j^i := \{p \in \mathbb{R}_+^n : -c_{ij}-p_j+p_i = \max\{-U+p_i, \max_k -c_{ik}-p_k+p_i\}\},$$
$$C_0^i := \{p \in \mathbb{R}_+^n : -c_{ij}-p_j+p_i \le -U+p_i \text{ for each } j \in N\}.$$

Then for every $i \in N$, $C^i = \{C_0^i, C_1^i, \ldots, C_n^i\}$ is a covering of $\mathbb{R}_+^n$. Further,

(i) The set C_j^i is closed for every $i,j \in N$.

(ii) The sets $C_1^i, C_2^i, \ldots, C_n^i$ cover the boundary of $\mathbb{R}_+^n$ for every $i \in N$.

(iii) There exists $L > 0$ such that if $p_j \ge L$, then $p \notin C_j^i$ for any j.

Hence, all conditions in Gale's theorem are satisfied and we have the existence of a price equilibrium.

The assignment game can be treated in a similar way leading to the same result.

4. BIPERMUTATION GAMES AND TRIDIMENSIONAL ASSIGNMENT GAMES

In this section we extend the assignment game and the permutation game to situations involving two types of indivisible goods. Every merchant in the extension of an assignment game has two indivisible goods of different types which he wants to sell and every buyer wants to buy one good of each type. Let b_j and c_j denote the values for merchant j of his good 1 and good 2 respectively. The value for buyer i of a bundle consisting of good 1 of merchant j and good 2 of merchant k we denote by h_{ijk}. We define $a_{ijk} = \max\{0, h_{ijk} - b_j - c_k\}$ and

$$v(S) := \max(a_{i_1 j_1 k_1} + \ldots + a_{i_p j_p k_p}) \text{ for every } S \in 2^N,$$

where $p = \min\{|S \cap B|, |S \cap M|\}$ and the maximum is taken over all assignments of pairs of players $(j_1,k_1),\ldots,(j_p,k_p)$ in $(S \cap M)\times(S \cap M)$ to players $i_1,\ldots,i_p$ in $S \cap B$. We call such a game a <u>tridimensional assignment game</u>. Contrary to the case with one indivisible good such a game not necessarily has a non-empty core, as we can see in the following theorem.

<u>Theorem</u>.
Let $|B| \geq 2$ and $|M| \geq 2$. Then there exists a tridimensional assignment game with an empty core.
<u>Proof</u>: First we consider the case $|B| = 2$ and $|M| = 2$. Let $B = \{1,2\}$, $B = \{3,4\}$ and $a_{133} = 2$, $a_{134} = a_{143} = 0$, $a_{144} = 2$ and $a_{233} = 0$, $a_{234} = 1$, $a_{243} = 3$, $a_{244} = 0$. Let $\langle B \cup M, v\rangle$ be the tridimensional assignment game defined by these a_{ijk}'s. Then $v(S) = 0$ for $S \subset B$ or $S \subset M$ and $v(\{1,3\}) = v(\{1,4\}) = v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = 2$, $v(\{2,3\}) = v(\{2,4\}) = 0$, $v(\{2,3,4\}) = v(\{1,2,3,4\}) = 3$. For $x = (x_1,x_2,x_3,x_4)$ to be in the core of this game x must satisfy: $x_i \geq 0$ for all $i \in \{1,2,3,4\}$, $x_1+x_3 \geq 2$, $x_1+x_4 \geq 2$, $x_2+x_3+x_4 \geq 3$ and $x_1+x_2+x_3+x_4 = 3$ or equivalently: $x_1 = 0$, $x_2 \geq 0$, $x_3 \geq 2$, $x_4 \geq 2$ and $x_1+x_2+x_3+x_4 = 3$. It is easy to see that the latter system has no solution and hence the core of the game is empty.
For $|B| > 2$ or $|M| > 2$ we obtain a tridimensional optimal assignment game with an empty core by taking the rest of the a_{ijk}'s to be equal to zero. □

<u>Remark</u>. It is straightforward to see that if $|B| = 1$ or $|M| = 1$ the allocation defined by giving the sole buyer or the sole merchant the amount v(N) and the others 0 is always an element

of the core.

In the extension of the permutation game every player has two indivisible goods. By c_{ijk} we denote the costs for player i associated with the bundle consisting of good 1 of player j and good 2 of player k. Instead of a cost matrix we have a cost cube $C = [c_{ijk}]^{n\ n\ n}_{i=1,j=1,k=1}$. We define

$$c(S) := \min_{\pi^1_S,\pi^2_S \in \Pi_S} \sum_{i \in S} c_{i\pi^1_S(i)\pi^2_S(i)} \quad \text{for every } S \in 2^N.$$

We call such a game a bipermutation game. In the definition of the game there occur two permutations, the first one describes how good 1 is redistributed among the players and the second one does the same for good 2.

In the following we give the cost cube C by writing down next to each other the layers with constant first index. Such a layer is an n×n-matrix, where the rows (columns) correspond to good 1 (good 2). The following theorem states that a bipermutation game need not have an empty core.

Theorem.

Let $|N| \geq 3$. Then there exists a bipermutation game $\langle N,c\rangle$ with an empty core.

Proof: First we consider the case that $N = \{1,2,3\}$ and where the cost cube C is given by

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Then $c(\{1\}) = c(\{2\}) = c(\{3\}) = c(\{1,2\}) = c(\{1,2,3\}) = 2$, $c(\{1,3\}) = 1$ and $c(\{2,3\}) = 0$. For $x = (x_1,x_2,x_3)$ to be an element of the core of this game x has to satisfy: $x_i \leq 2$ for each $i \in N$, $x_1+x_2 \leq 2$, $x_1+x_3 \leq 1$, $x_2+x_3 \leq 0$ and $x_1+x_2+x_3 = 2$ or equivalently: $x_1 = 2$, $x_2 \leq 0$, $x_2+x_3 = 0$ and $x_3 \leq -1$.

But the latter system has no solution and hence the core of the game is empty.

For $|N| > 3$ we get a game with an empty core by enlarging the cost cube C with zeroes. □

Remark. For $|N| = 2$ the core of any bipermutation game $\langle N,c\rangle$ is non-empty because the game is subadditive, and then each imputation is a core element.

The question arises why the proof that assignment games and permutation games have non-empty cores cannot be extended to the situation with two goods. We recall that an essential part of these proofs was the theorem of Birkhoff-von Neumann. As an extension of a doubly stochastic matrix we define a triply stochastic cube as follows:

Definition. A triply stochastic $n\times n\times n$-cube $T = [t_{ijk}]_{i=1,j=1,k=1}^{n\quad n\quad n}$ is an $n\times n\times n$-cube such that for all $i,j,k \in \{1,\dots,n\}$: $t_{ijk} \geq 0$ $\sum_{j,k} t_{ijk} = 1$, $\sum_{i,k} t_{ijk} = 1$, $\sum_{i,j} t_{ijk} = 1$.

So an $n\times n\times n$-cube is a triply stochastic cube if all its entries are nonnegative and the sums of all the layers are equal to 1. As an extension of a permutation matrix we define a bipermutation cube.

Definition. A $n\times n\times n$-bipermutation cube $Q = [q_{ijk}]_{i=1,j=1,k=1}^{n\quad n\quad n}$ is a triply stochastic cube with all entries q_{ijk} equal to zero or one.

It is easily seen that a bipermutation cube corresponds to an assignment of pairs of merchants to buyers in a tridimensional assignment game and to two permutations describing how the two goods are redistributed in a bipermutation game. Now if we want to extend the proofs for the cases with one good to the cases with two goods we need some result similar to the theorem of Birkhoff-von Neumann. This means that we should be able to write every triply stochastic cube as a convex combination of bipermutation matrices. That this is not the case is stated in the following theorem.

Theorem.

Let $n > 1$. Then the $n\times n\times n$-cube Z given by the n layers $\frac{1}{2}(E_{11}+E_{22}), \frac{1}{2}(E_{21}+E_{12}), E_{33}, E_{44}, \dots, E_{nn}$ where, for all $i,j \in N$, E_{ij} is the $n\times n$-matrix with a one in cell (i,j) and zeroes in the other cells, is a triply stochastic cube which cannot be written as a convex combination of bipermutation cubes.

Proof: It is clear that Z is a triply stochastic cube. Suppose $Z = \sum_{\ell=1}^{r} \alpha_\ell Q^\ell$, $\alpha_\ell > 0$ for $\ell \in \{1,\dots,r\}$, $\sum_{\ell=1}^{r} \alpha_\ell = 1$, $Q^\ell = [q^\ell_{ijk}]^n_{i=1,j=1,k=1}$ is a bipermutation cube for each $\ell \in \{1,\dots,r\}$.

Let $V_1 := \{\ell : q^\ell_{111} = 1\}$ and $V_2 := \{\ell : q^\ell_{122} = 1\}$. Then $V_1 \cap V_2 = \emptyset$ and $\sum_{\ell \in V_1} \alpha_\ell = \sum_{\ell=1}^{r} \alpha_\ell q^\ell_{111} = z_{111} = \frac{1}{2}$, $\sum_{\ell \in V_2} \alpha_\ell = \sum_{\ell=1}^{r} \alpha_\ell q^\ell_{122} = z_{122} = \frac{1}{2}$.

So $\sum_{\ell \in V_1 \cup V_2} \alpha_\ell = 1 = \sum_{\ell=1}^{r} \alpha_\ell$ which implies that $V_1 \cup V_2 = \{1,\dots,r\}$.

Now for $\ell \in V_1$ we have $q^\ell_{212} = 0$ because $q^\ell_{111} = 1$ and for $\ell \in V_2$ we have $q^\ell_{212} = 0$ because $q^\ell_{122} = 1$. But then $\sum_{\ell=1}^{r} \alpha_\ell q^\ell_{212} = 0 \neq \frac{1}{2} = z_{212}$. Hence, Z cannot be written as a convex combination of bipermutation cubes. □

In Curiel [1] all the 6 and 414 extreme points of the convex, compact set of triply stochastic 2×2×2-cubes, 3×3×3-cubes, respectively, are described.

5. SPECIAL CASES

In this section we describe two classes of tridimensional assignment games and bipermutation games for which the members have a non-empty core. The first class is the class of games with what we call additive revenues or costs.

Definition. We say that a tridimensional assignment game $\langle B \cup M, v \rangle$ has additive revenues if there exist real non-negative numbers a^1_{ij}, a^2_{ik} such that $a_{ijk} = a^1_{ij} + a^2_{ik}$ for all $i \in B$, $j,k \in M$.

A concrete situation in which additive revenues may arise is when the values of the two goods which the merchants want to sell are independent of each other. For example, the goods may be a house and a car. It is reasonable to assume that each car and each house has a value of its own for a buyer and that the value of a bundle consisting of a car and a house is just the sum of the separate values. That it is not always reasonable to assume that the revenues are additive we can see by considering the situation when the goods are guns and bullets. A gun has no value without bullets and bullets have no value without a gun. Further the value of a bundle consisting of a gun and bullets really depends on the combination of gun and bullets because not all bullets are suitable for all guns.

Definition. We say that a bipermutation game $\langle N,c\rangle$ has additive costs if there exist real numbers c^1_{ij}, c^2_{ik} such that $c_{ijk} = c^1_{ij}+c^2_{ik}$ for all $i,j,k \in N$.

A situation which may lead to a bipermutation game with additive costs is the following. Suppose every player needs to process a certain job for which he needs two types of machines. Every player possesses one machine of each type. The costs for processing a job are equal to the sum of the costs which arise when using each machine.

The following theorem concerns the non-emptiness of the core of these games.

Theorem.
Tridimensional assignment games with additive revenues and bipermutation games with additive costs have non-empty cores.

Proof: Let $\langle B \cup M,v\rangle$ be a tridimensional assignment game with additive revenues $a_{ijk} = a^1_{ij}+a^2_{ik}$ for all $i \in B$, $j,k \in M$. We define $\langle B \cup M,v^1\rangle$ and $\langle B \cup M,v^2\rangle$ to be the assignment games with revenues a^1_{ij}, $i \in B$, $j \in M$, a^2_{ik}, $i \in B$, $k \in M$, respectively. Then $v(S) = v^1(S)+v^2(S)$ for every $S \in 2^N$ and v^1 and v^2 have non-empty cores by Shapley-Shubik's result. It is straightforward to see that $C(v) \supset C(v_1)+C(v_2)$ and hence $C(v) \neq \emptyset$. The proof that a bipermutation game with additive costs has a non-empty core is similar to this proof. □

The second class we want to consider is the class of games with so-called separable revenues or costs.

Definition. A bipermutation game $\langle N,c\rangle$ is said to have separable costs if there exist real numbers d_i, e_{jk} such that $c_{ijk} = d_i+e_{jk}$ for all $i,j,k \in N$.

An example with separable costs is the following. Suppose that the players have to process a job on two machines. The costs depend on the combination of machines used. Every job done on a certain combination (j,k) of machines incurs the same cost e_{jk}. But before player i can start doing his job on the machine he has to transport resources to the machines. This incurs the cost d_i. So his total costs are d_i+e_{jk}.

<u>Definition</u>. A tridimensional assignment game $\langle B \cup M, v\rangle$ is said to have <u>separable revenues</u> if there exist real numbers $b_i, f_{jk} \geq 0$ such that $a_{ijk} = b_i+f_{jk}$ for all $i \in B$, $j,k \in M$.

For an example of separable revenues we go back to our guns and bullets merchants. The value f_{jk} of a certain combination (j,k) of gun and bullets depends only on this combination. The factor b_i we consider to be a number which describes how much the possession of any combination of gun and bullets is worth to buyer i.

Now we prove that these game also have non-empty cores.

<u>Theorem</u>.
Bipermutation games with separable costs and tridimensional assignment games with separable revenues have non-empty cores.

<u>Proof</u>: Let $\langle N,c\rangle$ be a bipermutation game with separable costs $c_{ijk} = d_i+e_{jk}$ for all $i,j,k \in N$. We define $\langle N,c'\rangle$ to be the permutation game with cost matrix $[e_{jk}]_{j,k\in N}$. Then $c(S) = c'(S)+ \sum_{i\in S} d_i$ for every $S \in 2^N$ and c' has a non-empty core by the theorem of Tijs et al. [6]. It is straightforward to see that if $(x_1,\ldots,x_n) \in C(c')$ then $(x_1+d_1,\ldots,x_n+d_n) \in C(c)$ and hence $C(c) \neq \emptyset$. In a similar way we can prove that a tridimensional optimal assignment game with separable revenues has a non-empty core. □

<u>Remark</u>. As a generalization of the additive case the proof of the non-emptiness of the core can be extended to the case with $c_{ijk} = g(c^1_{ij}, c^2_{ik})$ for all $i,j,k \in N$ where the function $g : \mathbb{R}^2 \to \mathbb{R}$ is of the form $g(x,y) = \lambda_1 x+\lambda_2 y+\lambda_3$ with $\lambda_1, \lambda_2 \geq 0$.

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